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Some results on Fredholm and semi-Fredholm perturbations

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Abstract In this paper, we investigate some properties of semi-Fredholm operators on Banach spaces. These results are applied to the determination of the stability of various essential spectra of closed densely defined linear operators. Also, we generalize some results in the literature and we extend and unify those obtained in Jeribi (J Math Anal Appl 271:343–358, 2002), Jeribi (J Math Anal Appl 275:222–237, 2002), Jeribi (Ser Math Inf 17:35–55, 2002), Jeribi (Arch Inequal Appl 2:123–140, 2004), Latrach and Dehici (J Math Anal Appl 259:277–301, 2001), Latrach and Paoli (J Aust Math Soc 77:73–89, 2004).

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المخلص

نبحث في هذه الورقة بعض خصائص نصف مؤثرات فيريدولم على فضاءات باناخ. تطبق تلك النتائج لتحديد استقرار عدة أطياف جوهرية لمؤثرات خطية مغلقة معرفة بشكل كثيف. نعمم أيضاً بعض النتائج الموجودة، ونمدد ونوحد تلك التي تم الحصول عليها في [27, 24, 19, 18, 17, 16].

1 Introduction

Let X and Y be two Banach spaces. By an operator A from X into Y we mean a linear operator with domain $\mathcal{D}(A) \subseteq X$ and range contained in Y . We denote by $\mathcal{C}(X, Y)$ (resp., $\mathcal{L}(X, Y)$) the set of all closed, densely defined (resp., bounded) linear operators from X to Y . The subset of all compact (resp., weakly compact) operators of $\mathcal{L}(X, Y)$ is designated by $\mathcal{K}(X, Y)$ (resp., $\mathcal{W}(X, Y)$). If $A \in \mathcal{C}(X, Y)$, we write $N(A) \subset X$ and $R(A) \subset Y$ for the null space and the range of A . We set $\alpha := \dim N(A)$ and $\beta := \text{codim } R(A)$. Let $A \in \mathcal{C}(X, Y)$ with closed range. Then A is a Φ_+ -operator ($A \in \Phi_+(X, Y)$) if $\alpha(A) < \infty$, and then A is a Φ_- -operator ($A \in \Phi_-(X, Y)$) if $\beta(A) < \infty$. $\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y)$ is the class of Fredholm operators while $\Phi_{\pm}(X, Y)$ denotes the set $\Phi_{\pm}(X, Y) = \Phi_+(X, Y) \cup \Phi_-(X, Y)$. For $A \in \Phi(X, Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathcal{L}(X, Y)$, $\mathcal{K}(X, Y)$, $\mathcal{W}(X, Y)$, $\mathcal{C}(X, Y)$, $\Phi_+(X, Y)$, $\Phi_{\pm}(X, Y)$ and $\Phi(X, Y)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{W}(X)$, $\mathcal{C}(X)$, $\Phi_+(X)$, $\Phi_{\pm}(X)$ and $\Phi(X)$. Let $A \in \mathcal{C}(X)$, the spectrum of A will be denoted by $\sigma(A)$. The resolvent set of A , $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in Φ_{+A} , Φ_{-A} , $\Phi_{\pm A}$ or Φ_A if $\lambda - A$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_{\pm}(X)$ or $\Phi(X)$, respectively. For the properties of these sets we refer to [5, 8, 21, 31].

It is well known that if A is a bounded self-adjoint operator on a Hilbert space, the essential spectrum $\sigma_{\text{ess}}(A)$ is the set of all points of the spectrum of A that are not isolated eigenvalues of finite algebraic multiplicity (see,

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for example [11, 29, 33]). Irrespective of whether A is bounded or not on a Banach space X , there are several definitions of the essential spectrum, most are enlargement of the continuous spectrum. Define the sets

$$\begin{aligned}\sigma_{e1}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_+(X)\} := \mathbb{C} \setminus \Phi_+(A), \\ \sigma_{e2}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_-(X)\} := \mathbb{C} \setminus \Phi_-(A), \\ \sigma_{e3}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_\pm(X)\} := \mathbb{C} \setminus \Phi_\pm(A), \\ \sigma_{e4}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi(X)\} := \mathbb{C} \setminus \Phi(A), \\ \sigma_{e5}(A) &:= \mathbb{C} \setminus \rho_5(A), \\ \sigma_{e6}(A) &:= \mathbb{C} \setminus \rho_6(A),\end{aligned}$$

where $\rho_5(A) := \{\lambda \in \Phi_A; i(\lambda - A) = 0\}$ and $\rho_6(A) := \{\lambda \in \rho_5(A) \text{ such that all scalars near } \lambda \text{ are in } \rho(A)\}$. $\sigma_{e1}(\cdot)$ and $\sigma_{e2}(\cdot)$ are the Gustafson and Weidman essential spectra [10]. $\sigma_{e3}(\cdot)$ is the Kato essential spectrum [22]. $\sigma_{e4}(\cdot)$ is the Wolf essential spectrum [10, 30, 33]. $\sigma_{e5}(\cdot)$ is the Schechter essential spectrum [10, 30, 31] and $\sigma_{e6}(\cdot)$ denotes the Browder essential spectrum [10, 30]. Note that all these sets are closed and, in general, we have

$$\sigma_{e1}(A) \cap \sigma_{e2}(A) = \sigma_{e3}(A) \subseteq \sigma_{e4}(A) \subseteq \sigma_{e5}(A) \subseteq \sigma_{e6}(A).$$

But if X is a Hilbert space and A is self-adjoint, then all these sets coincide.

One of the central questions in the study of the essential spectra of closed densely defined operator A on Banach space X consists of showing what are the conditions that we must impose on $K \in \mathcal{C}(X)$ in order that $\sigma_{ei}(A + K) = \sigma_{ei}(A)$, $i = 1, \dots, 6$. In this direction some authors have been interested by this question concerning the stability of the Schechter essential spectrum and they have proved the following:

If K is a strictly singular operator on L_p -spaces $p \geq 1$, then $\sigma_{e5}(A + K) = \sigma_{e5}(A)$ (see [26, Theorem 3.2]). If K is a weakly compact operator on Banach spaces which possess the Dunford–Pettis property (see Definition 2.2), then $\sigma_{e5}(A + K) = \sigma_{e5}(A)$ (see [23, Theorem 3.2]). If $K \in \mathcal{L}(X)$ such that $(\lambda - A)^{-1}K$ is a strictly singular (resp., weakly compact) operator on L_p -spaces $p > 1$ (resp., on Banach spaces which possess the Dunford–Pettis property), then $\sigma_{e5}(A + K) = \sigma_{e5}(A)$ (see [14, 15]). In [16] Jeribi extended this analysis to the case of general Banach spaces where a detailed treatment of the Schechter essential spectrum of a closed densely defined linear operator A subjected to additive perturbations K such that $(\lambda - A)^{-1}K$ or $K(\lambda - A)^{-1}$ belonging to arbitrary subsets of $\mathcal{L}(X)$ contained in the ideal of Fredholm perturbations. In [17, 18] Jeribi extended these results to unbounded perturbations. His approach consists principally in considering the class of A -closable operator K (not necessarily bounded) which contained in the set of A -resolvent Fredholm perturbations, and of proving that $\sigma_{e5}(A + K) = \sigma_{e5}(A)$ for all $K \in \mathcal{C}(X)$ such that K is A -bounded and $K(\lambda - A)^{-1} \in \mathcal{J}(X)$ for some $\lambda \in \rho(A)$ or $K(\lambda - A - K)^{-1} \in \mathcal{J}(X)$ for all $\lambda \in \rho(A + K)$ where $\mathcal{J}(X) = \{K \in \mathcal{L}(X) \text{ such that } I - K \in \Phi(X) \text{ and } i(I - K) = 0\}$. Recently, in [19] this author has extended the analysis used above for the Wolf essential spectrum, the Schechter essential spectrum and the Browder essential spectrum. More precisely, let X be a Banach space and let $A \in \mathcal{C}(X)$. Let $\mathcal{I}(X)$ be an arbitrary two-sided ideal of $\mathcal{L}(X)$ satisfying the condition $\mathcal{F}_0(X) \subset \mathcal{I}(X) \subset \mathcal{J}(X)$, where $\mathcal{F}_0(X)$ stands for the ideal of finite rank operators. Then $\sigma_{ei}(A) = \sigma_{ei}(A + J)$, $i = 4, 5$ for all $J \in \mathcal{I}(X)$ and if $C\sigma_{e5}(A)$ [the complement of $\sigma_{e5}(A)$] is connected and neither $\rho(A)$ nor $\rho(A + J)$ is empty, then $\sigma_{e6}(A) = \sigma_{e6}(A + J)$. The purpose of this work is to pursue the analysis started in [12, 13, 16–20, 24–27] and to extend it to general Banach spaces.

We organize the paper in the following way : the next section is devoted to some Fredholm perturbation results and stability of the essential spectra of closed densely defined linear operators on a Banach space.

2 Perturbation results

In the beginning of this section we introduce some definitions.

Definition 2.1 Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. F is called a Fredholm perturbation if $U + F \in \Phi(X, Y)$ whenever $U \in \Phi(X, Y)$. F is called an upper (resp., lower) Fredholm perturbation if $U + F \in \Phi_+(X, Y)$ (resp., $U + F \in \Phi_-(X, Y)$) whenever $U \in \Phi_+(X, Y)$ (resp., $U \in \Phi_-(X, Y)$).

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X, Y)$, $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$, respectively. In general, we have

$$\begin{aligned}\mathcal{K}(X, Y) &\subseteq \mathcal{F}_+(X, Y) \subseteq \mathcal{F}(X, Y) \\ \mathcal{K}(X, Y) &\subseteq \mathcal{F}_-(X, Y) \subseteq \mathcal{F}(X, Y).\end{aligned}$$



If $X = Y$ we write $\mathcal{F}(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X, X)$, $\mathcal{F}_+(X, X)$ and $\mathcal{F}_-(X, X)$, respectively. Let $\Phi^b(X, Y)$, $\Phi_+^b(X, Y)$ and $\Phi_-^b(X, Y)$ denote the sets $\Phi(X, Y) \cap \mathcal{L}(X, Y)$, $\Phi_+(X, Y) \cap \mathcal{L}(X, Y)$ and $\Phi_-(X, Y) \cap \mathcal{L}(X, Y)$, respectively. If in Definition 1.1 we replace $\Phi(X, Y)$, $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$ by $\Phi^b(X, Y)$, $\Phi_+^b(X, Y)$ and $\Phi_-^b(X, Y)$ we obtain the sets $\mathcal{F}^b(X, Y)$, $\mathcal{F}_+^b(X, Y)$ and $\mathcal{F}_-^b(X, Y)$. These classes of operators were introduced and investigated in [6]. In particular, it is shown that $\mathcal{F}^b(X, Y)$ is a closed subset of $\mathcal{L}(X, Y)$ and $\mathcal{F}^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general we have

$$\begin{aligned}\mathcal{K}(X, Y) &\subseteq \mathcal{F}_+^b(X, Y) \subseteq \mathcal{F}^b(X, Y) \\ \mathcal{K}(X, Y) &\subseteq \mathcal{F}_-^b(X, Y) \subseteq \mathcal{F}^b(X, Y).\end{aligned}$$

In this paper, we prove that the two sets $\mathcal{F}^b(X, Y)$ and $\mathcal{F}(X, Y)$ are equal. In fact, the inclusion $\mathcal{F}(X, Y) \subset \mathcal{F}^b(X, Y)$ is evident. The other inclusion is based on some results of Fredholm theory which are found in [31]. It must be noted that the problem of the equalities between the sets $\mathcal{F}_+^b(X, Y)$ and $\mathcal{F}_-^b(X, Y)$, respectively, with the sets $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$ remains open.

Definition 2.2 Let X be a Banach space. X is said to have the Dunford–Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y .

The Dunford–Pettis property as defined above was explicitly defined by Grothendieck [9] who undertook an extensive study of this and related properties. It is well known that any L_1 space has the property DP [3]. Also, if \emptyset is a compact Hausdorff space $C(\emptyset)$ has the property DP [9]. For further examples, we refer to [1] or [4, pp. 494, 479, 508 and 511]. Note that the property DP is not conserved under conjugation. However, if X is a Banach space whose dual has the property DP then X has the property DP (see, e.g., [9]). For more information, we refer to the paper by Diestel [2] which contains a survey and exposition of the Dunford–Pettis property and related topics.

Remark 2.3 It is proved in [23, Proposition 3.1] that if X is a Banach space with the property DP, then

$$\mathcal{W}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X).$$

Let $A \in \mathcal{C}(X, Y)$, the graph norm of A is defined by

$$\|x\|_A = \|x\| + \|Ax\|, \quad x \in \mathcal{D}(A)$$

where $\mathcal{D}(A)$ denotes the domain of A . It follows from the closedness of A that $\mathcal{D}(A)$ endowed with the norm $\|\cdot\|_A$ is a Banach space. In this new space, denoted by X_A the operator A satisfies $\|Ax\| \leq \|x\|_A$ and consequently, $A \in \mathcal{L}(X_A, X)$. Let $J : X \longrightarrow Y$ be a linear operator on X . If $\mathcal{D}(A) \subset \mathcal{D}(J)$, then J will be called A -defined. If J is A -defined, we will denote by \widehat{J} its restriction to $\mathcal{D}(A)$. Moreover, if $\widehat{J} \in \mathcal{L}(X_A, Y)$, we say that J is A -bounded. One checks easily that if J is closed (or closable) (cf. [22, Remark 1.5, p. 191]), then J is A -bounded.

Definition 2.4 We say that an operator J is A -closed if $x_n \rightarrow x$, $Ax_n \rightarrow y$, $Jx_n \rightarrow z$ for $\{x_n\} \subseteq \mathcal{D}(A)$ implies that $x \in \mathcal{D}(J)$ and $Jx = z$. It will be called A -closable if $x_n \rightarrow 0$, $Jx_n \rightarrow z$ implies $z = 0$.

Remark 2.5 (i) If J is bounded, then J is A -bounded.
(ii) If J is closed, then J is A -closed.
(iii) If J is closable, then J is A -closable.
(iv) If A is closed, then by [30, Lemma 2.1], we get J is A -closed if and only if J is A -closable if and only if J is A -bounded.

Let J be an arbitrary A -bounded operator. Hence, we can regard A and J as operators from X_A into Y . They will denoted by \widehat{A} and \widehat{J} , respectively. These belong to $\mathcal{L}(X_A, Y)$. Furthermore, we have obvious relations

$$\begin{cases} \alpha(\widehat{A}) = \alpha(A), \quad \beta(\widehat{A}) = \beta(A), \quad R(\widehat{A}) = R(A), \\ \alpha(\widehat{A} + \widehat{J}) = \alpha(A + J), \\ \beta(\widehat{A} + \widehat{J}) = \beta(A + J) \text{ and } R(\widehat{A} + \widehat{J}) = R(A + J). \end{cases}$$



Definition 2.6 Let X be a Banach space, $A \in \mathcal{C}(X)$ and F be an arbitrary A -defined linear operator on X . We say that F is an A -Fredholm perturbation if $\widehat{F} \in \mathcal{F}^b(X_A, X)$. F is called an upper (resp., lower) A -Fredholm perturbation if $\widehat{F} \in \mathcal{F}_+^b(X_A, X)$ (resp., $\widehat{F} \in \mathcal{F}_-^b(X_A, X)$).

Let $A\mathcal{F}(X)$, $A\mathcal{F}_+(X)$ and $A\mathcal{F}_-(X)$ designate the sets of A -Fredholm, upper A -Fredholm and lower A -Fredholm perturbations, respectively.

Definition 2.7 Let X and Y be two Banach spaces, $A \in \mathcal{C}(X, Y)$ and let $F : X \longrightarrow Y$ be an arbitrary A -defined linear operator. We say that F is an unbounded A -Fredholm perturbation if $\widehat{F} \in \mathcal{F}(X_A, Y)$. F is called an unbounded upper (resp., unbounded lower) A -Fredholm perturbation if $\widehat{F} \in \mathcal{F}_+(X_A, Y)$ (resp., $\widehat{F} \in \mathcal{F}_-(X_A, Y)$).

Let $UA\mathcal{F}(X, Y)$, $UA\mathcal{F}_+(X, Y)$ and $UA\mathcal{F}_-(X, Y)$ designate the sets of unbounded A -Fredholm, unbounded upper A -Fredholm and unbounded lower A -Fredholm perturbations, respectively.

If $X = Y$ we write $UA\mathcal{F}(X)$, $UA\mathcal{F}_+(X)$ and $UA\mathcal{F}_-(X)$ for $UA\mathcal{F}(X, X)$, $UA\mathcal{F}_+(X, X)$ and $UA\mathcal{F}_-(X, X)$, respectively.

Definition 2.8 Let $A \in \mathcal{C}(X, Y)$ and let $J : X \longrightarrow Y$ be an arbitrary A -defined linear operator. We say that J is A -compact if $\widehat{J} \in \mathcal{K}(X_A, Y)$.

Let $A\mathcal{K}(X, Y)$ denote the set of all linear operators from X to Y which are A -compact on X .

Remark 2.9 (i) Clearly, if J is a compact operator, then J is A -compact operator. (ii) A consequence of Definition 1.5 and the inclusions (1.1) and (1.2) that

$$\begin{aligned} A\mathcal{K}(X, Y) &\subseteq UA\mathcal{F}_+(X, Y) \subseteq UA\mathcal{F}(X, Y), \\ A\mathcal{K}(X, Y) &\subseteq UA\mathcal{F}_-(X, Y) \subseteq UA\mathcal{F}(X, Y). \end{aligned}$$

Proposition 2.10 Let X, Y and Z be three Banach spaces.

(i) If the set $\Phi^b(Y, Z)$ is not empty, then

$$\begin{aligned} E_1 \in \mathcal{F}_+^b(X, Y) \text{ and } A \in \Phi^b(Y, Z) &\text{ imply } AE_1 \in \mathcal{F}_+^b(X, Z) \\ E_1 \in \mathcal{F}_-^b(X, Y) \text{ and } A \in \Phi^b(Y, Z) &\text{ imply } AE_1 \in \mathcal{F}_-^b(X, Z). \end{aligned}$$

(ii) If the set $\Phi^b(X, Y)$ is not empty, then

$$\begin{aligned} E_2 \in \mathcal{F}_+^b(Y, Z) \text{ and } B \in \Phi^b(X, Y) &\text{ imply } E_2B \in \mathcal{F}_+^b(X, Z) \\ E_2 \in \mathcal{F}_-^b(Y, Z) \text{ and } B \in \Phi^b(X, Y) &\text{ imply } E_2B \in \mathcal{F}_-^b(X, Z). \end{aligned}$$

Proof (i) Since $A \in \Phi^b(Y, Z)$, it follows by [31, Theorem 1.4, p. 108] that there exist $A_0 \in \mathcal{L}(Z, Y)$ and $K \in \mathcal{K}(Z)$ such that $AA_0 = I - K$. From [26, Lemma 2.2] we get $AA_0 \in \Phi^b(Z)$. By [31, Theorem 3.4, p. 117] we have $A_0 \in \Phi^b(Z, Y)$. Let $J \in \Phi_+^b(X, Z)$ (resp., $\Phi_-^b(X, Z)$), using [28, Theorem 5, p. 150], we deduce that $A_0J \in \Phi_+^b(X, Y)$ (resp., $\Phi_-^b(X, Y)$). This implies that $(E_1 + A_0J) \in \Phi_+^b(X, Y)$ (resp., $\Phi_-^b(X, Y)$). So, $A(E_1 + A_0J) \in \Phi_+^b(X, Z)$ (resp., $\Phi_-^b(X, Z)$). Next, using the relation

$$AE_1 + J - KJ = A(E_1 + A_0J)$$

together with the compactness of the operator KJ we get $(AE_1 + J) \in \Phi_+^b(X, Z)$ (resp., $\Phi_-^b(X, Z)$).

This implies that $AE_1 \in \mathcal{F}_+^b(X, Z)$ (resp., $\mathcal{F}_-^b(X, Z)$).

(ii) The proof of (ii) is obtained as like as the proof of (i). \square



Theorem 2.11 *Let X, Y and Z be three Banach spaces.*

(i) *If the set $\Phi^b(Y, Z)$ is not empty, then*

$$\begin{aligned} E_1 \in \mathcal{F}_+^b(X, Y) \quad \text{and} \quad A \in \mathcal{L}(Y, Z) \quad \text{imply} \quad AE_1 \in \mathcal{F}_+^b(X, Z) \\ E_1 \in \mathcal{F}_-^b(X, Y) \quad \text{and} \quad A \in \mathcal{L}(Y, Z) \quad \text{imply} \quad AE_1 \in \mathcal{F}_-^b(X, Z). \end{aligned}$$

(ii) *If the set $\Phi^b(X, Y)$ is not empty, then*

$$\begin{aligned} E_2 \in \mathcal{F}_+^b(Y, Z) \quad \text{and} \quad B \in \mathcal{L}(X, Y) \quad \text{imply} \quad E_2B \in \mathcal{F}_+^b(X, Z) \\ E_2 \in \mathcal{F}_-^b(Y, Z) \quad \text{and} \quad B \in \mathcal{L}(X, Y) \quad \text{imply} \quad E_2B \in \mathcal{F}_-^b(X, Z). \end{aligned}$$

Proof (i) Let $C \in \Phi^b(Y, Z)$ and $\lambda \in \mathbb{C}$. Setting $A_1 = A - \lambda C$ and $A_2 = \lambda C$. For λ sufficiently large, using [31, Theorem 3.2, p. 115] we have $A_1 \in \Phi^b(Y, Z)$. It follows from Proposition 2.10 (i) that $A_1E_1 \in \mathcal{F}_+^b(X, Z)$ (resp., $\mathcal{F}_-^b(X, Z)$) and $A_2E_1 \in \mathcal{F}_+^b(X, Z)$ (resp., $\mathcal{F}_-^b(X, Z)$). This implies $A_1E_1 + A_2E_1 = AE_1$ is an element of $\mathcal{F}_+^b(X, Z)$ (resp., $\mathcal{F}_-^b(X, Z)$).
(ii) The proof may be sketched in a similar way to (i), it suffices to replace Proposition 2.10 (i) by Proposition 2.10 (ii). \square

Corollary 2.12 *Let X be a Banach space. Then $\mathcal{F}_-^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.*

Proposition 2.13 *Let X, Y and Z be three Banach spaces. If $\Phi^b(Y, Z)$ is not empty, then*

$$E_1 \in \mathcal{F}_+(X, Y) \quad \text{and} \quad A \in \Phi^b(Y, Z) \quad \text{imply} \quad AE_1 \in \mathcal{F}_+(X, Z).$$

Proof Since $A \in \Phi^b(Y, Z)$, it follows by [31, Theorem 1.4, p. 108] that there exist $A_0 \in \mathcal{L}(Z, Y)$ and a finite rank operator K on Z such that $AA_0 = I - K$. Using [26, Lemma 2.2] we have $AA_0 \in \Phi^b(Z)$. By [31, Theorem 3.4, p. 117] we get $A_0 \in \Phi_+^b(Z, Y)$. Let $J \in \Phi_+(X, Z)$. Since $\mathcal{D}(A_0J) = \mathcal{D}(J)$ is dense in X , then by [31, Theorem 6.4, p. 184] we have $A_0J \in \Phi_+(X, Y)$. This implies that $(E_1 + A_0J) \in \Phi_+(X, Y)$. So, $A(E_1 + A_0J) \in \Phi_+(X, Z)$.

We claim that KJ is $(AE_1 + J)$ -compact. Indeed, let $x \in \mathcal{D}(J)$, we have

$$\begin{aligned} \|Jx\| &= \|(AE_1 + J)x - AE_1x\| \\ &\leq \|(AE_1 + J)x\| + \|AE_1x\| \\ &\leq \|(AE_1 + J)x\| + \|A\| \|E_1\| \|x\| \\ &\leq \max(1, \|A\| \|E_1\|) (\|(AE_1 + J)x\| + \|x\|). \end{aligned}$$

Hence, using the last inequality we have

$$\begin{aligned} \|KJ(x)\| &\leq \|K\| \|Jx\| \\ &\leq \max(1, \|A\| \|E_1\|) \|K\| (\|(AE_1 + J)x\| + \|x\|). \end{aligned}$$

So, KJ is $(AE_1 + J)$ -compact, which proves the claim. Next, using the relation

$$AE_1 + J - KJ = A(E_1 + A_0J),$$

KJ being $(AE_1 + J)$ -compact and refereeing to [32, Theorem 2.12, p. 9] and [22, Theorem 5.26, p. 238], one sees that $(AE_1 + J) \in \Phi_+(X, Z)$. This implies that $AE_1 \in \mathcal{F}_+(X, Z)$ and completes the proof. \square

Proposition 2.14 [7, Proposition 3.2, p. 374] *Let X, Y and Z be three Banach spaces, $T : X \longrightarrow Y$ be a closed operator with closed range and $\dim N(T) < \infty$ and $C : Z \longrightarrow X$ be a closed operator. Then TC is a closed operator.*

Proposition 2.15 *Let X, Y and Z be three Banach spaces, $T \in \mathcal{C}(X, Y)$ and $S \in \mathcal{C}(Z, X)$. If $T \in \Phi^b(X, Y)$ and $S \in \Phi_-(Z, X)$, then $TS \in \Phi_-(Z, Y)$.*



Proof By Proposition 2.14 the operator TS is closed. Put $N_1 = R(S) \cap N(T)$. Since $N(T)$ is finite dimensional,

$$N(T) = N_1 \oplus N_2,$$

for some finite dimensional subspace N_2 . Obviously, $R(S) \cap N_2 = \{0\}$. Furthermore $R(S) \oplus N_2$ is closed, because $R(S)$ is closed and $\dim N_2 < \infty$ (see [31, Lemma 2.1, p. 107]). Next, we prove that there exists a finite dimensional subspace N_3 such that

$$(R(S) \oplus N_2) \oplus N_3 = X, \quad N_3 \subset \mathcal{D}(T) \quad (2.1)$$

put $X_0 = R(S) \oplus N_2$ and let $k = \dim X/X_0$. Note that $k \leq \text{codim } R(S) < \infty$. If $k = 0$, then we take $N_3 = \{0\}$ in (2.1). Assume $k > 0$. Since $\overline{\mathcal{D}(T)} = X$ and X_0 is closed, $\mathcal{D}(T)$ is not entirely contained in X_0 . So there exists a vector $x_1 \in \mathcal{D}(T)$ such that $x_1 \notin X_0$. Put $X_1 = X_0 \oplus \text{span}\{x_1\}$. Then X_1 is closed and $\dim X/X_1 = k - 1$. Thus we can repeat the above reasoning for X_1 in place of X_0 . Proceeding in this way, we find k steps vectors x_1, \dots, x_k in $\mathcal{D}(T)$ such that $X = X_0 \oplus \text{span}\{x_1, \dots, x_k\}$. Put $N_3 = \text{span}\{x_1, \dots, x_k\}$ and (2.1) is fulfilled. The space N_3 is isomorphic to the quotient space $R(T)/R(TS)$ under the map $u \rightarrow [Tu]$, because of (2.1). Indeed, if $x \in \mathcal{D}(T)$, then (2.1) implies that $x = Sz + v + u$, where $z \in \mathcal{D}(S)$, $v \in N_2 \subset N(T)$ and $u \in N_3$. It follows that $Sz = x - v - u \in \mathcal{D}(T)$ and $T(Sz) = Tx - Tu$, which shows that $[Tx] = [Tu]$. Furthermore, if $[Tu] = [0]$ for $u \in N_3$, then

$$u \in R(S) + N(T) = R(S) \oplus N_2,$$

and hence $u = 0$. So $u \mapsto [Tu]$ has the desired properties, and thus

$$\beta(TS) = \beta(T) + \dim N_3. \quad (2.2)$$

Now (2.2) shows that $R(TS)$ is closed (see [7, p. 372]). \square

Proposition 2.16 *Let X, Y and Z be three Banach spaces. If $\Phi^b(Y, Z)$ is not empty, $E_1 \in \mathcal{F}_-(X, Y)$ and $A \in \Phi^b(Y, Z)$ then $AE_1 \in \mathcal{F}_-(X, Z)$.*

Proof Since $A \in \Phi^b(Y, Z)$, it follows by [31, Theorem 1.4, p. 108] that there exist $A_0 \in \mathcal{L}(Z, Y)$ and a finite rank operator K on Z such that $AA_0 = I - K$. Using [26, Lemma 2.2] we have $AA_0 \in \Phi^b(Z)$. By [31, Theorem 3.4, p. 117] we get $A_0 \in \Phi^b(Z, Y)$ and so $A_0 \in \Phi_+^b(Z, Y)$. Let $J \in \Phi_-(X, Z)$. By Proposition 2.15 we have $A_0J \in \Phi_-(X, Y)$. This implies that $(E_1 + A_0J) \in \Phi_-(X, Y)$. So, $A(E_1 + A_0J) \in \Phi_-(X, Z)$. Now arguing as in the proof of Proposition 2.13, we prove that KJ is $(AE_1 + J)$ -compact. Next, using the relation

$$AE_1 + J - KJ = A(E_1 + A_0J),$$

KJ being $(AE_1 + J)$ -compact and refereeing to [32, Theorem 2.12, p. 9] and [22, Theorem 5.26, p. 238], one sees that $(AE_1 + J) \in \Phi_-(X, Z)$. This implies that $AE_1 \in \mathcal{F}_-(X, Z)$. \square

Theorem 2.17 *Let X, Y and Z be three Banach spaces. If $\Phi^b(Y, Z)$ is not empty, then*

$$\begin{aligned} A \in \mathcal{L}(Y, Z) \quad \text{and} \quad E_1 \in \mathcal{F}_+(X, Y) \quad \text{imply} \quad AE_1 \in \mathcal{F}_+(X, Z) \\ A \in \mathcal{L}(Y, Z) \quad \text{and} \quad E_1 \in \mathcal{F}_-(X, Y) \quad \text{imply} \quad AE_1 \in \mathcal{F}_-(X, Z). \end{aligned}$$

Proof Let $C \in \Phi^b(Y, Z)$ and $\lambda \in \mathbb{C}$. Setting $A_1 = A - \lambda C$ and $A_2 = \lambda C$. For λ sufficiently large, using [31, Theorem 3.2, p. 115] we have $A_1 \in \Phi^b(Y, Z)$. It follows from Proposition 2.13 (resp., Proposition 2.16) that $A_1E_1 \in \mathcal{F}_+(X, Z)$ (resp., $\mathcal{F}_-(X, Z)$) and $A_2E_1 \in \mathcal{F}_+(X, Z)$ (resp., $\mathcal{F}_-(X, Z)$). This implies $A_1E_1 + A_2E_1 = AE_1$ is an element of $\mathcal{F}_+(X, Z)$ (resp., $\mathcal{F}_-(X, Z)$). \square

Proposition 2.18 *Let X, Y and Z be three Banach spaces. If $\Phi^b(Y, Z)$ is not empty, $E_1 \in \mathcal{F}(X, Y)$ and $A \in \Phi^b(Y, Z)$, then $AE_1 \in \mathcal{F}(X, Z)$.*

Proof Since $A \in \Phi^b(Y, Z)$, it follows by [31, Theorem 1.4 p. 108] that there exist $A_0 \in \mathcal{L}(Z, Y)$ and a finite rank operator K on Z such that $AA_0 = I - K$. Using [26, Lemma 2.2] we have $AA_0 \in \Phi^b(Z)$. By [31, Theorem 3.4, p. 117] we get $A_0 \in \Phi^b(Z, Y)$. Let $J \in \Phi(X, Z)$. By [31, Theorem 1.3, p. 163] we have $A_0J \in \Phi(X, Y)$. This implies that $(E_1 + A_0J) \in \Phi(X, Y)$. So, $A(E_1 + A_0J) \in \Phi(X, Z)$. Now arguing as in the proof of Proposition 2.13, we prove that KJ is $(AE_1 + J)$ -compact. Next, using the relation

$$AE_1 + J - KJ = A(E_1 + A_0J),$$

KJ being $(AE_1 + J)$ -compact and refereeing to [32, Theorem 2.12, p. 9] and [31, Theorem 2.3, p. 168], one sees that $(AE_1 + J) \in \Phi(X, Z)$. This implies that $AE_1 \in \mathcal{F}(X, Z)$. \square



Theorem 2.19 Let X, Y and Z be three Banach spaces. If $\Phi^b(Y, Z)$ is not empty, then

$$A \in \mathcal{L}(Y, Z) \text{ and } E_1 \in \mathcal{F}(X, Y) \text{ imply } AE_1 \in \mathcal{F}(X, Z).$$

Proof Let $C \in \Phi^b(Y, Z)$ and $\lambda \in \mathbb{C}$. Setting $A_1 = A - \lambda C$ and $A_2 = \lambda C$. For λ sufficiently large, using [31, Theorem 3.2, p. 115] we have $A_1 \in \Phi^b(Y, Z)$. It follows from Proposition 2.18 that $A_1 E_1 \in \mathcal{F}(X, Z)$ and $A_2 E_1 \in \mathcal{F}(X, Z)$. This implies $A_1 E_1 + A_2 E_1 = AE_1$ is an element of $\mathcal{F}(X, Z)$. \square

Theorem 2.20 Let X and Y be two Banach spaces. Then

$$\mathcal{F}^b(X, Y) = \mathcal{F}(X, Y).$$

Remark 2.21 Note that for $X = Y$, Theorem 2.20 is nothing but Lemma 2.3 (ii) in [24].

Proof of Theorem 2.4 Clearly $\mathcal{F}(X, Y) \subset \mathcal{F}^b(X, Y)$ (because $\Phi^b(X, Y) \subseteq \Phi(X, Y)$). To prove the opposite inclusion, let $F \in \mathcal{F}^b(X, Y)$. If $A \in \Phi(X, Y)$, then by [31, Theorem 1.1, p. 162] there exist $A_0 \in \mathcal{L}(Y, X)$ and $K \in \mathcal{L}(Y)$, of finite rank such that

$$AA_0 = I - K \text{ on } Y. \quad (2.3)$$

Thus

$$(A + F)A_0 = I - K + FA_0 = I + E. \quad (2.4)$$

By [31, Corollary 1.6, p. 166], the fact that $A \in \Phi(X, Y)$ implies that $\hat{A} \in \Phi^b(X_A, Y)$. Also, (2.3) implies that AA_0 is a Fredholm operator. Next, applying [31, Theorem 2.7, p. 171] we obtain that $A_0 \in \Phi^b(Y, X_A)$. Similarly, since $A_0 \in \mathcal{L}(Y, X)$, we conclude using [6, pp. 69-70], that $E \in \mathcal{F}^b(Y)$. This together with (2.4) implies that $(A + F)A_0 \in \Phi^b(Y)$. Since $A_0 \in \Phi^b(Y, X_A)$, it follows from [31, Theorem 2.5, p. 169] that $\hat{A} + \hat{F} \in \Phi^b(X_A, Y)$. Now by [31, Lemma 1.7, p. 166] we see that $A + F \in \Phi(X, Y)$. This shows that $F \in \mathcal{F}(X, Y)$ which ends the proof. \square

Corollary 2.22 Let X be a Banach space and $A \in \mathcal{C}(X)$, then $UAF(X) = AF(X)$. Then

Proposition 2.23 Let X and Y be two Banach spaces, $A \in \mathcal{C}(X, Y)$. The following statements are satisfied.

- (i) $\mathcal{F}(X, Y) \subset UAF(X, Y)$.
- (ii) $\mathcal{F}_+(X, Y) \subset UAF_+(X, Y)$.
- (iii) $\mathcal{F}_-(X, Y) \subset UAF_-(X, Y)$.

Proof (i) Let $F \in \mathcal{F}(X, Y)$ and $B \in \Phi(X_A, Y)$. Since, X_A is continuously embedded in X and is dense in X , we have by [31, Lemma 1.7, p. 166], $B \in \Phi(X, Y)$. So $F + B \in \Phi(X, Y)$. By [31, Corollary 1.6, p. 166], we get $\hat{F} + B \in \Phi(X_A, Y)$. Hence, $F \in UAF(X_A, Y)$.

The proof of (ii) and (iii) may be sketched in a similar way to (i). \square

The following lemma generalizes Lemma 3.1 in [27].

Lemma 2.24 Let $A \in \mathcal{C}(X, Y)$ and let $J : X \longrightarrow Y$ be a linear operator. Assume that $J \in UAF(X, Y)$. Then

- (i) if $A \in \Phi(X, Y)$, then $A + J \in \Phi(X, Y)$ and $i(A + J) = i(A)$. Moreover,
- (ii) if $A \in \Phi_+(X, Y)$ and $J \in UAF_+(X, Y)$, then $A + J \in \Phi_+(X, Y)$,
- (iii) if $A \in \Phi_-(X, Y)$ and $J \in UAF_-(X, Y)$, then $A + J \in \Phi_-(X, Y)$,
- (iv) if $A \in \Phi_\pm(X, Y)$ and $J \in UAF_+(X, Y) \cap UAF_-(X, Y)$, then $A + J \in \Phi_\pm(X, Y)$.

Proof Since $A \in \mathcal{C}(X, Y)$ and $J \in UAF(X, Y)$, hence as mentioned above we can regard A and J as operators from X_A into Y . They will be denoted by \hat{A} and \hat{J} , respectively. These belong to $\mathcal{L}(X_A, Y)$ and we have

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \beta(\hat{A}) = \beta(A), R(\hat{A}) = R(A), \\ \alpha(\hat{A} + \hat{J}) = \alpha(A + J), \\ \beta(\hat{A} + \hat{J}) = \beta(A + J) \text{ and } R(\hat{A} + \hat{J}) = R(A + J). \end{cases} \quad (2.5)$$



Observe that the assertion (ii), (iii) and (iv) are immediate. (i) Assume that $A \in \Phi(X, Y)$. Then using (2.5) we infer that $\widehat{A} \in \Phi^b(X_A, Y)$. Hence it follows from [31, Theorem 1.4 p. 108] that there $A_0 \in \mathcal{L}(Y, X_A)$ and $K \in \mathcal{K}(X_A)$ such that:

$$A_0 \widehat{A} = I - K. \quad (2.6)$$

This leads to

$$\begin{aligned} A_0(\widehat{A} + \widehat{J}) &= I - K + A_0 \widehat{J} \\ &= I - Q. \end{aligned} \quad (2.7)$$

Next, it follows from (2.6) that $A_0 \widehat{A} \in \Phi^b(X_A)$ and $i(A_0 \widehat{A}) = 0$. Hence the use of [31, Theorem 3.4, p. 117] together with the Atkinson theorem [31, Theorem 2.3, p. 111] implies that $A_0 \in \Phi^b(Y, X_A)$ and

$$i(\widehat{A}) = -i(A_0). \quad (2.8)$$

On the other hand, since $\widehat{J} \in UAF(X, Y)$ and $A_0 \in \mathcal{L}(Y, X_A)$, applying Theorem 2.19 we get $A_0 \widehat{J} \in \mathcal{F}(X_A)$. Using the fact that $\mathcal{K}(X_A) \subset \mathcal{F}(X_A)$ we infer that $Q \in \mathcal{F}(X_A)$. Therefore applying Proposition 3.1 (i) in [24] to (2.7), we get $A_0(\widehat{A} + \widehat{J}) \in \Phi^b(X_A)$ and $i[A_0(\widehat{A} + \widehat{J})] = 0$. Since $A_0 \in \Phi^b(Y, X_A)$, it follows from [31, Theorem 3.4, p. 117] and the Atkinson theorem that $(\widehat{A} + \widehat{J}) \in \Phi^b(X_A, Y)$ and

$$i(\widehat{A} + \widehat{J}) = -i(A_0). \quad (2.9)$$

Now using Eqs. (2.5), (2.8) and (2.9) we find that $i(A + J) = i(A)$ which completes the proof. \square

Remark 2.25 (i) Note that the result of Lemma 2.24 (iii) remains true if we suppose that Y is a reflexive space, (because if Y is not a reflexive space, we do not guarantee that A^* is densely defined) and if we consider $J^* \in UAF_+(Y^*, X^*)$. In fact, let $A \in \Phi_-(X, Y)$. Applying [31, Theorem 4.1, p.177] and [22, Theorem 5.13, p. 234] we infer that $A^* \in \Phi_+(Y^*, X^*)$. Moreover, $J^* \in UAF_+(Y^*, X^*)$ implies that $A^* + J^* \in \Phi_+(Y^*, X^*)$. This together with the fact that $\alpha(A^* + J^*) = \beta(A + J)$ (use again [22, Theorem 5.13, p. 234]) gives the result.

(ii) Note that for $X = Y$, Lemma 2.24 (i) is nothing but Lemma 3.1 (i) in [27] when we use Corollary 2.22.

Theorem 2.26 Let $A \in \mathcal{C}(X)$ and J be an operator on X . The following statements are satisfied.

(i) If $J \in UAF(X)$, then

$$\sigma_{ei}(A) = \sigma_{ei}(A + J), \quad i = 4, 5.$$

Moreover, if $\mathbb{C} \setminus \sigma_{e5}(A)$ [the complement of $\sigma_{e5}(A)$] is connected and neither $\rho(A)$ nor $\rho(A + J)$ is empty, then

$$\sigma_{e6}(A) = \sigma_{e6}(A + J).$$

Further,

(ii) if $J \in UAF_+(X)$, then

$$\sigma_{e1}(A) = \sigma_{e1}(A + J),$$

(iii) if $J \in UAF_-(X)$, then

$$\sigma_{e2}(A) = \sigma_{e2}(A + J),$$

(iv) if $J \in UAF_+(X) \cap UAF_-(X)$, then

$$\sigma_{e3}(A) = \sigma_{e3}(A + J).$$

Proof The proofs of the items (ii), (iii), (iv) and the first part of (i) for $i = 4$ use Lemma 2.24 and are immediate. So, they are omitted. The proof of (i) for $i = 5$ is similar to (i) for $i = 5$ of Theorem 2.11 in [27]. \square

Let X be a Banach space and $A \in \mathcal{C}(X)$. The point (resp., residual, continuous) spectrum of A will be denoted by $\sigma_p(A)$ (resp., $\sigma_r(A)$, $\sigma_c(A)$).



Corollary 2.27 *Let X be a Banach space and $A \in \mathcal{C}(X)$. The following statements are satisfied.*

(i) *If $\sigma_{e5}(A)$ or $\sigma_{e6}(A)$ is empty, then for every $J \in UAF(X)$ we have :*

$$\sigma(A + J) = \sigma_p(A + J).$$

(ii) *If $\sigma_{e1}(A)$ is empty, then for every $J \in UAF_+(X)$ we have :*

$$\sigma(A + J) = \sigma_p(A + J) \cup \sigma_r(A + J).$$

(iii) *If $\sigma_{e2}(A)$ is empty, then for every $J \in UAF_-(X)$ we have :*

$$\sigma(A + J) = \sigma_p(A + J) \cup \sigma_r(A + J).$$

(iv) *If $\sigma_{e3}(A)$ is empty, then for every $J \in UAF_+(X) \cap UAF_-(X)$ we have :*

$$\sigma(A + J) = \sigma_p(A + J) \cup \sigma_r(A + J).$$

(v) *If $\sigma_{e4}(A)$ is empty, then for every $J \in UAF(X)$ we have :*

$$\sigma(A + J) = \sigma_p(A + J) \cup \sigma_r(A + J).$$

Proof This corollary follows immediately from Theorem 2.26 and the fact that $\sigma_c(A) \subseteq \bigcap_{i=1}^6 \sigma_{ei}(A)$ and $\sigma_r(A) \subseteq \sigma_{e5}(A) \subseteq \sigma_{e6}(A)$. \square

In general, Theorem 2.26 can not be directly used in applications. So, we give in the following theorem practical criteria which guarantee the invariance of the different essential spectra by unbounded A_- , upper unbounded A_- and lower unbounded A -Fredholm perturbations.

Theorem 2.28 *Let $A, B \in \mathcal{C}(X)$ and $\lambda \in \rho(A) \cap \rho(B)$. The following statements are satisfied.*

(i) *If $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in UAF(X)$, then*

$$\sigma_{ei}(A) = \sigma_{ei}(B), \quad i = 4, 5.$$

Further,

(ii) *if $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in UAF_+(X)$, then*

$$\sigma_{e1}(A) = \sigma_{e1}(B),$$

(iii) *if $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in UAF_-(X)$, then*

$$\sigma_{e2}(A) = \sigma_{e2}(B),$$

(iv) *if $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in UAF_+(X) \cap UAF_-(X)$, then*

$$\sigma_{e3}(A) = \sigma_{e3}(B).$$

Proof Without loss of generality, we may assume that $\lambda = 0$. Hence $0 \in \rho(A)$ and therefore

$$\mu - A = -\mu(\mu^{-1} - A^{-1})A, \quad \mu \neq 0.$$

Since A is one to one and onto, then $\alpha(\mu - A) = \alpha(\mu^{-1} - A^{-1})$ and $R(\mu - A) = R(\mu^{-1} - A^{-1})$. This shows that $\mu \in \Phi_{+A}$ (resp., Φ_{-A}) if and only if $\mu^{-1} \in \Phi_{+A^{-1}}$ (resp., $\Phi_{-A^{-1}}$). Similarly we have $\mu \in \Phi_A$ if and only if $\mu^{-1} \in \Phi_{A^{-1}}$. (i) If $A^{-1} - B^{-1} \in UAF(X)$, then using Lemma 2.24 (i) we conclude that $\Phi_A = \Phi_B$ and $i(\mu - A) = i(\mu - B)$ for all $\mu \in \Phi_A$. (ii) and (iii) If $A^{-1} - B^{-1} \in UAF_+(X)$ (resp., $UAF_-(X)$), then using Lemma 2.24 (ii) (resp., Lemma 2.24 (iii)) we conclude that $\Phi_{+A} = \Phi_{+B}$ (resp., $\Phi_{-A} = \Phi_{-B}$). This concludes the proof of (ii) (resp., (iii)). (iv) Using (ii), (iii) and Lemma 2.24 (iv) we have

$$\Phi_{-A} \cup \Phi_{+A} = \Phi_{-B} \cup \Phi_{+B}.$$

This ends the proof. \square



Corollary 2.29 Let $A \in \mathcal{C}(X)$ and let J be an A -bounded operator on X , and assume that there is a complex number $\lambda \in \rho(A)$ such that $r_\sigma(J(\lambda - A)^{-1}) < 1$. Then :

(i) If $J(\lambda - A)^{-1} \in UAF(X)$, then

$$\sigma_{ei}(A) = \sigma_{ei}(A + J), \quad i = 4, 5.$$

Moreover,

(ii) if $J(\lambda - A)^{-1} \in UAF_+(X)$ then,

$$\sigma_{e1}(A) = \sigma_{e1}(A + J).$$

(iii) if $J(\lambda - A)^{-1} \in UAF_-(X)$ then,

$$\sigma_{e2}(A) = \sigma_{e2}(A + J).$$

(iv) if $J(\lambda - A)^{-1} \in UAF_+(X) \cap UAF_-(X)$ then, $\sigma_{e3}(A) = \sigma_{e3}(A + J)$.

Proof Let $\lambda \in \rho(A)$. Since J is A -bounded, according to Lemma 2.24 in [30], $J(\lambda - A)^{-1}$ is a closed linear operator defined on all X and therefore bounded by the closed graph theorem. On the other hand, the assumption $r_\sigma(J(\lambda - A)^{-1}) < 1$ implies that $\lambda \in \rho(A + J)$ and

$$(\lambda - A - J)^{-1} - (\lambda - A)^{-1} = \sum_{n \geq 1} (\lambda - A)^{-1} [J(\lambda - A)^{-1}]^n.$$

Clearly, if $J(\lambda - A)^{-1} \in UAF(X)$ (resp., $UAF_+(X)$, $UAF_-(X)$, $UAF_+(X) \cap UAF_-(X)$), then the closedness of $UAF(X)$ (resp., $UAF_+(X)$, $UAF_-(X)$, $UAF_+(X) \cap UAF_-(X)$) and the use of Theorem 2.19 (resp., 2.2) imply that $(\lambda - A - J)^{-1} - (\lambda - A)^{-1} \in UAF(X)$ (resp., $UAF_+(X)$, $UAF_-(X)$, $UAF_+(X) \cap UAF_-(X)$). Now the items (i), (ii), (iii) and (iv) follow immediately from Theorem 2.28. \square

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